THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics

MATH 2050A Tutorial 5

1. (a) Let $I := (a, b]$, where $a < b$. Show that the set of cluster points of I is $[a, b]$.
   (b) Write down the set of cluster points of the following subsets of $\mathbb{R}$:
      i. $\{x_1, x_2, \cdots, x_N\}$
      ii. $\mathbb{N}$
      iii. $\{1/n : n \in \mathbb{N}\}$
      iv. $I \cap \mathbb{Q}$,where $I = [0, 1]$

2. Use the definition of limit to show that

$$\lim_{x \to -1} \frac{x^2 + 2x + 4}{x + 2} = 3$$

3. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$. Suppose c is cluster point of A.
   (a) Prove the **Sequential Criterion for limit** : $\lim_{x \to c} f(x) = l$ $(\in \mathbb{R})$ if and only if every sequence (x_n) in $A \backslash \{c\}$ converging to c, $(f(x_n))$ converges to l.
   (b) Suppose $\lim_{x \to c} f(x)$ does not exist. Show that there exists $\epsilon_o > 0$ and two sequences (x_n) and (y_n) in $A \backslash \{c\}$, both converging to c, such that $|f(x_n) - f(y_n)| \geq \epsilon_o$ for all $n \in \mathbb{N}$.
   (c) Prove the **Cauchy Criterion for limit** : $\lim_{x \to c} f(x)$ exists if and only if for all $\epsilon > 0$, there exists $\delta > 0$ such that if $x, y \in A$ with $0 < |x - c|, |y - c| < \delta$, then $|f(x) - f(y)| < \epsilon$.

4. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) := \begin{cases} x & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

   (a) Show that f has a limit at $x = 0$..
   (b) Show that if $c \neq 0$, then f does not have a limit at c.